

HOMOLOGY OF PRECROSSED MODULES

DANIEL ARIAS, MANUEL LADRA, AND ALFREDO R-GRANDJEÁN

ABSTRACT. We prove that the category of precrossed modules is an algebraic category. This leads to a cotriple (co)homology theory for precrossed modules, which generalizes the Eilenberg-MacLane (co)homology groups and we study their relationship with the (co)homology of crossed modules.

1. INTRODUCTION

From a certain point of view precrossed modules are generalizations of groups and they form a model of homotopy type in dimensions 1 and 2 for connected CW-complexes. The precrossed modules of nilpotence class 2 play an essential role in Baues's algebraic models of homotopy 3-types [3].

Conduché and Ellis define in [7] the first and second homology group of a precrossed P -module, and in [11] Inassaridze and Khmaladze extend this theory defining all the homology groups modulo an integer q for a precrossed P -module, in terms of nonabelian derived functors, where P is a fixed action group.

In this paper we develop the basic elements to define cotriple homology and cohomology theories in the whole category of precrossed modules.

We start with the observation that the category of precrossed modules is an algebraic category, that is, there exists a tripleable forgetful functor from that category to the category of sets. The (co)homology theory for precrossed modules is therefore a particularization of Barr and Beck's general theory [1].

We will make a general study of this cotriple (co)homology theory specially concerned to the relationship with other classical or new (co)homology theories. It is shown that this theory generalizes the Eilenberg-MacLane (co)homology groups if we think a group G as a precrossed module $(1, G, i)$ or $(G, 1, 1)$. On the other hand, in the particular case that we take a crossed module (T, G, ∂) we get the low dimensional homology for crossed modules

Date: September 6, 2001.

1991 Mathematics Subject Classification. Primary 18G60, 18G30; Secondary 20J05.

Key words and phrases. homology, precrossed module, simplicial resolution.

Work partially supported by PGIDT01PXI20702PR and by MCYT, project BFM2000-0523. Spain.

of Gilbert [9], and their generalization in every dimension. Finally the relationship with Carrasco, Cegarra and R.-Grandjeán's CCG -(co)homology of crossed modules [5] is made through a long exact sequence connecting both theories.

We begin in section 2 by recalling standard facts about the category of precrossed modules. Then we demonstrate that this category is algebraic, and so we generalize a result from [5] about tripleability for crossed modules. At this point new concepts of free and projective precrossed modules arise.

In section 3 we define the cotriple (co)homology theory for precrossed modules. Its basic properties are shown.

Finally, in section 4 we work out the relationships with other (co)homology theories described above.

2. PRECROSSED MODULES ARE AN ALGEBRAIC CATEGORY

A *precrossed module* (M, P, μ) is a group homomorphism $\mu : M \longrightarrow P$ together with an action of P on M , denoted ${}^p m$ for $p \in P$ and $m \in M$, satisfying $\mu({}^p m) = p\mu(m)p^{-1}$ for all $p \in P$ and $m \in M$. If in addition it verifies the Peiffer's identity $\mu({}^{\mu(m)} m') = mm'm'^{-1}$ for all $m, m' \in M$ it is said that (M, P, μ) is a *crossed module*.

Examples.

(i) Let N be a normal subgroup of G . The inclusion homomorphism $i : N \longrightarrow G$ with the action ${}^g n = gng^{-1}$, $g \in G, n \in N$, is a crossed module. In particular, (G, G, Id) and $(1, G, i)$ are crossed modules.

(ii) $(A, G, 0)$ is a crossed module where A is an ordinary ZG -module.

(iii) Let P, G be groups and P acting on G non trivially or G non-abelian group, then $(G \rtimes P, P, \pi)$ is a precrossed module but not a crossed module, where $G \rtimes P$ is the semidirect product of G and P , π is the natural surjective homomorphism of $G \rtimes P$ on P and the action of P on $G \rtimes P$ is ${}^{p'}(g, p) = (p'g, p')$. In particular, if G is a non-abelian group, $(G, 1, 1)$ is a precrossed module but not a crossed module.

A *precrossed module morphism* $(\Phi, \Psi) : (M_1, P_1, \mu_1) \longrightarrow (M_2, P_2, \mu_2)$ is a pair of group homomorphisms $\Phi : M_1 \longrightarrow M_2$ and $\Psi : P_1 \longrightarrow P_2$ such that $\Psi \circ \mu_1 = \mu_2 \circ \Phi$ and $\Phi({}^p m) = {}^{\Psi(p)} \Phi(m)$ for all $p \in P_1$ and $m \in M_1$.

We denote the category of precrossed modules by $\mathcal{PCM}$.

A morphism (Φ, Ψ) in $\mathcal{PCM}$ is said to be *injective* (*surjective*) if both Φ and Ψ are injective (surjective) group homomorphisms.

A *precrossed submodule* (N, Q, μ') of a precrossed module (M, P, μ) is a precrossed module such that N and Q are respectively subgroups of M and P , the action of Q on N is induced by the one of P on M and $\mu|_N = \mu'$. It is said to be a *normal precrossed submodule* if besides N and Q are normal in M and P , ${}^p n \in N$ and ${}^q mm^{-1} \in N$ for all $p \in P$, $q \in Q$, $m \in M$ and $n \in N$.

If (N, Q, μ) is a normal precrossed submodule of (M, P, μ) , we define the *quotient precrossed module* $(M, P, \mu)/(N, Q, \mu)$ as $(M/N, P/Q, \bar{\mu})$ where the homomorphism $\bar{\mu}$ is induced by μ and P/Q acts on M/N by ${}^pQmN = ({}^pm)N$ for $p \in P$ and $m \in M$.

We call the *Peiffer subgroup* $\langle M, M \rangle$ of a precrossed module (M, P, μ) to be the subgroup of M generated by the Peiffer elements $m_1 m_2 m_1^{-1} \mu(m_1) m_2^{-1}$ with $m_1, m_2 \in M$. It is a normal subgroup of M , and the quotient precrossed module $(M, P, \mu)/(\langle M, M \rangle, 1, 1)$ is a crossed module termed the *Peiffer abelianisation* of (M, P, μ) .

The *kernel* of a precrossed module morphism $(\Phi, \Psi) : (M_1, P_1, \mu_1) \rightarrow (M_2, P_2, \mu_2)$ is the normal precrossed submodule $(\text{Ker}\Phi, \text{Ker}\Psi, \mu_1)$ of (M_1, P_1, μ_1) . Its *image* is the precrossed submodule $(\text{Im}\Phi, \text{Im}\Psi, \mu_2)$ of (M_2, P_2, μ_2) .

If two precrossed modules (M_1, P_1, μ_1) and (M_2, P_2, μ_2) are given, their *product* is the precrossed module $(M_1 \times M_2, P_1 \times P_2, \mu_1 \times \mu_2)$, with $P_1 \times P_2$ acting on $M_1 \times M_2$ by $({}^{p_1, p_2})(m_1, m_2) = ({}^{p_1}m_1, {}^{p_2}m_2)$ for $p_1 \in P_1, p_2 \in P_2, m_1 \in M_1$ and $m_2 \in M_2$.

Now we will introduce analogues of some basic concepts from group theory, like centre or commutator groups, in the category of precrossed modules.

We call the *centre* $Z(M, P, \mu)$ of a precrossed module (M, P, μ) to be the normal precrossed submodule $(\text{Inv}(M) \cap Z(M), \text{St}_P(M) \cap Z(P), \mu)$, where $\text{St}_P(M)$ denotes the group $\{p \in P \mid {}^p m = m \text{ for all } m \in M\}$, $\text{Inv}(M) = \{m \in M \mid \mu(m) \in \text{St}_P(M) \text{ and } {}^p m = m \text{ for all } p \in P\}$ and $Z(M), Z(P)$ denote the centres of M and P . By the same proof as given in [13], $Z(M, P, \mu)$ is the maximal central precrossed submodule of (M, P, μ) and so $Z(M, P, \mu)$ coincides with the categorical notion of centre developed by Huq in [10].

Thus a precrossed module (M, P, μ) is said to be *abelian* if $(M, P, \mu) = Z(M, P, \mu)$. Equivalently M and P are abelian groups and P acts trivially on M . The variety of abelian precrossed modules $\mathcal{APCM}$ is exactly the one of abelian crossed modules. We can identify it with the category of homomorphisms of abelian groups which is equivalent to the category of right modules over the ring of matrices $\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ [5].

If (N, Q, μ) is a normal precrossed submodule of (M, P, μ) , we define the *commutator precrossed submodule* $[(N, Q, \mu), (M, P, \mu)]$ of (N, Q, μ) and (M, P, μ) as the normal precrossed submodule $([N, M] [Q, M] [P, N], [P, Q], \mu)$ of (M, P, μ) , where $[P, N]$ denotes the subgroup of M generated by the elements $\{{}^p n n^{-1} \mid p \in P, n \in N\}$, $[Q, M]$ denotes the subgroup of M generated by the elements $\{{}^q m m^{-1} \mid q \in Q, m \in M\}$ and $[N, M]$ and $[P, Q]$ denote the usual commutator subgroups of N with M and P with Q .

In particular, the *commutator precrossed submodule* $[(M, P, \mu), (M, P, \mu)]$ of a precrossed module (M, P, μ) is the normal precrossed submodule

$([M, M][P, M], [P, P], \mu)$ of (M, P, μ) . It is the smallest precrossed module of (M, P, μ) making the quotient an abelian precrossed module.

To prove the tripleability of $\mathcal{PCM}$ over $\mathcal{Set}$ we will use the following criterion, due to Linton [8]. For the background on tripleability we will refer to [2].

THEOREM 2.1. *A functor $U : D \longrightarrow \mathcal{Set}$ is tripleable if and only if U has a left adjoint and the following three conditions are satisfied:*

- (a) *D has kernel pairs and coequalizers*
- (b) *$p : Y \longrightarrow Z$ is a coequalizer $\iff Up : UY \longrightarrow UZ$ is a coequalizer*
- (c) *$X \xrightarrow[s]{t} Y$ is a kernel pair $\iff UX \xrightarrow[U]{Us} UY$ is a kernel pair, where p, s and t denote morphisms in D .*

The criterion will be applied to the forgetful functor $U : \mathcal{PCM} \longrightarrow \mathcal{Set}$, $U(M, P, \mu) = M \times P$, that assigns to each precrossed module (M, P, μ) the cartesian product of the underlying sets M and P .

PROPOSITION 2.2. *The functor $U : \mathcal{PCM} \longrightarrow \mathcal{Set}$ has a left adjoint.*

Proof. It is known [12] that the forgetful functor

$$U_1 : \mathcal{PCM} \longrightarrow \mathcal{Grp} \downarrow \mathcal{Grp}, \quad U_1(M, P, \mu) = \mu$$

has a left adjoint by the functor $F_1 : \mathcal{Grp} \downarrow \mathcal{Grp} \longrightarrow \mathcal{PCM}$, $F_1(G \xrightarrow{\lambda} H) = (\overline{G}, H, \langle \lambda, Id \rangle_{|\overline{G}})$, where $\overline{G} = \text{Ker}(G * H \xrightarrow{(0, Id)} H)$, $\langle \lambda, Id \rangle : G * H \longrightarrow H$ and H acts on $\overline{G}$ by conjugation.

On the other hand, the functor

$$U_2 : \mathcal{Grp} \downarrow \mathcal{Grp} \longrightarrow \mathcal{Grp}, \quad U_2(G \xrightarrow{\lambda} H) = G \times H$$

has a left adjoint by the functor $F_2 : \mathcal{Grp} \longrightarrow \mathcal{Grp} \downarrow \mathcal{Grp}$, taking a group G into $F_2(G) = (G \xrightarrow{i_1} G * G)$ the first inclusion in the coproduct.

We denote by $U_3 : \mathcal{Grp} \longrightarrow \mathcal{Set}$ the usual forgetful functor and by $F_3 : \mathcal{Set} \longrightarrow \mathcal{Grp}$ its left adjoint, the free group functor.

Composing the three described adjunctions

$$\mathcal{Set} \xrightleftharpoons[U_3]{F_3} \mathcal{Grp} \xrightleftharpoons[U_2]{F_2} \mathcal{Grp} \downarrow \mathcal{Grp} \xrightleftharpoons[U_1]{F_1} \mathcal{PCM}$$

we obtain that $U = U_3 \circ U_2 \circ U_1 : \mathcal{PCM} \longrightarrow \mathcal{Set}$ is right adjoint to $\mathcal{F} = F_1 \circ F_2 \circ F_3 : \mathcal{Set} \longrightarrow \mathcal{PCM}$, given by $\mathcal{F}(X) = (\overline{F}, F * F, \langle i_1, Id \rangle_{|\overline{F}})$,

where $F = F_3(X)$ is the free group over X , $\overline{F} = \text{Ker}(F * (F * F) \xrightarrow{\langle 0, Id \rangle} F * F)$, $\langle i_1, Id \rangle : F * (F * F) \longrightarrow F * F$ and $F * F$ acts on $\overline{F}$ by conjugation. $\square$

THEOREM 2.3. *The functor $\mathcal{U} : \mathcal{PCM} \longrightarrow \mathcal{Set}$ is tripleable.*

Proof. We check the conditions of Theorem 2.1:

(a) $\mathcal{PCM}$ has kernel pairs. It is easy to check that the kernel pair of a precrossed module morphism $(M_1, P_1, \mu_1) \longrightarrow (M_2, P_2, \mu_2)$ is the precrossed submodule $(M_1 \times_{M_2} M_1, P_1 \times_{P_2} P_1, \mu_1 \times \mu_1)$ of the product $(M_1, P_1, \mu_1) \times (M_1, P_1, \mu_1)$, with the projections to the first and second components

$$(M_1 \times_{M_2} M_1, P_1 \times_{P_2} P_1, \mu_1 \times \mu_1) \xrightarrow[\begin{smallmatrix} (\pi_M^2, \pi_P^2) \end{smallmatrix}]{\begin{smallmatrix} (\pi_M^1, \pi_P^1) \end{smallmatrix}} (M_1, P_1, \mu_1)$$

$\mathcal{PCM}$ has coequalizers. Given a pair of morphisms

$$(M_1, P_1, \mu_1) \xrightarrow[\begin{smallmatrix} (\Psi_M, \Psi_P) \end{smallmatrix}]{\begin{smallmatrix} (\Phi_M, \Phi_P) \end{smallmatrix}} (M_2, P_2, \mu_2)$$

consider the least normal precrossed submodule (N, Q, μ_2) of (M_2, P_2, μ_2) such that the canonical projection $(\pi_N, \pi_Q) : (M_2, P_2, \mu_2) \longrightarrow (M_2, P_2, \mu_2) / (N, Q, \mu_2)$ satisfies $(\pi_N, \pi_Q) \circ (\Phi_M, \Phi_P) = (\pi_N, \pi_Q) \circ (\Psi_M, \Psi_P)$. It is possible to get (N, Q, μ_2) since the intersection (in the obvious way) of normal precrossed submodules satisfying the condition is a normal precrossed submodule that also verifies it. It is clear that (π_N, π_Q) is the coequalizer of (Φ_M, Φ_P) and (Ψ_M, Ψ_P) .

(b) $\mathcal{U}$ preserves and reflects coequalizers because both in $\mathcal{PCM}$ as in $\mathcal{Set}$, the coequalizers are the surjective morphisms (a surjective precrossed module morphism is the coequalizer of its kernel inclusion and the zero map).

(c) $\mathcal{U}$ preserves kernel pairs because it has a left adjoint.

To see that $\mathcal{U}$ reflects kernel pairs, first note that by a *congruence* on a precrossed module (M, P, μ) we shall mean a precrossed submodule $(R, C, \mu \times \mu)$ of the product precrossed module $(M, P, \mu) \times (M, P, \mu) = (M \times M, P \times P, \mu \times \mu)$ such that both $R \subset M \times M$ and $C \subset P \times P$ are equivalence relations. As in groups, giving a congruence $(R, C, \mu \times \mu)$ on a precrossed module (M, P, μ) is equivalent to giving a normal precrossed submodule $(N_R, Q_C, \mu) \triangleleft (M, P, \mu)$, where $N_R = \{m \in M \mid (1, m) \in R\}$ and $Q_C = \{p \in P \mid (1, p) \in C\}$. In fact

$$(R, C, \mu \times \mu) \xrightarrow[\begin{smallmatrix} (\pi_M^2, \pi_P^2) \end{smallmatrix}]{\begin{smallmatrix} (\pi_M^1, \pi_P^1) \end{smallmatrix}} (M, P, \mu) \text{ is the kernel pair of the quotient } (M, P, \mu) \longrightarrow (M, P, \mu) / (N_R, Q_C, \mu).$$

Take a pair $(M_1, P_1, \mu_1) \xrightarrow[(\Psi_M, \Psi_P)]{(\Phi_M, \Phi_P)} (M_2, P_2, \mu_2)$ of precrossed module morphisms which is a kernel pair in $\mathcal{Set}$. This implies that $(\Phi_M \times \Phi_P) \times (\Psi_M \times \Psi_P) : M_1 \times P_1 \longrightarrow (M_2 \times P_2) \times (M_2 \times P_2)$ is injective, and its image is an equivalence relation on the group $M_2 \times P_2$. Thus $(\Phi_M, \Phi_P) \times (\Psi_M, \Psi_P) : (M_1, P_1, \mu_1) \longrightarrow (M_2, P_2, \mu_2) \times (M_2, P_2, \mu_2)$ is an injective morphism that establishes a precrossed module isomorphism between (M_1, P_1, μ_1) and its image, denoted by $(R, C, \mu_2 \times \mu_2)$. It is clear that $(R, C, \mu_2 \times \mu_2)$ is a congruence on (M_2, P_2, μ_2) .

The result is straightforward since this isomorphism makes commutative the triangles

$$\begin{array}{ccc}
 (M_1, P_1, \mu_1) & \xrightarrow[(\Psi_M, \Psi_P)]{(\Phi_M, \Phi_P)} & (M_2, P_2, \mu_2) \\
 & \searrow & \uparrow (\pi_M^1, \pi_P^1) \quad \uparrow (\pi_M^2, \pi_P^2) \\
 & & (R, C, \mu_2 \times \mu_2)
 \end{array} \quad \square$$

Observe that every Birkhoff subvariety of $\mathcal{PCM}$ (closed under subobjects, quotients and products) is closed for constructing kernel pairs and coequalizers too, so by Theorem 2.1 it will be also algebraic. As a corollary we get a theorem of [5]:

COROLLARY 2.4. *The category of crossed modules $\mathcal{CM}$ is tripleable over $\mathcal{Set}$, via the forgetful functor $\nu : \mathcal{CM} \longrightarrow \mathcal{Set}$, $\nu(T, G, \partial) = T \times G$.*

The existence of a good pair of adjoint functors between $\mathcal{PCM}$ and $\mathcal{Set}$ will help us to study some useful facts on free and projective precrossed modules.

Remember that an object P in a category is said to be *projective* if for every regular epimorphism (i.e. a coequalizer) $p : A \twoheadrightarrow B$, the function $p_* : \text{Hom}(P, A) \longrightarrow \text{Hom}(P, B)$ is a surjective map. A category is said to have *enough projective objects* if for each object Y , there exists a regular epimorphism $P \twoheadrightarrow Y$ with P a projective object (that is, a *projective presentation* of Y).

As it was seen in Theorem 2.3, regular epimorphisms in $\mathcal{PCM}$ are just surjective morphisms. So for each set X , the free precrossed module (relative to U) $\mathcal{F}(X)$ is a projective precrossed module, and every precrossed module (M, P, μ) has a projective presentation through the counit of the adjunction $\mathcal{FU}(M, P, \mu) \twoheadrightarrow (M, P, \mu)$.

There are more projective objects in $\mathcal{PCM}$. In the following proposition we give a family of projective objects with some members not isomorphic to any value of $\mathcal{F}$.

PROPOSITION 2.5. *Let P and Q be free groups. The precrossed module $(\overline{P}, P * Q, \langle i_P, Id \rangle_{|\overline{P}})$ is projective, where $\overline{P} = \text{Ker}(P * (P * Q) \xrightarrow{\langle 0, Id \rangle} P * Q)$, $\langle i_P, Id \rangle : P * (P * Q) \longrightarrow P * Q$ and $P * Q$ acts on $\overline{P}$ by conjugation.*

Proof. The inclusion in the coproduct $i_P : P \rightarrowtail P * Q$ is a projective object in the category of group homomorphisms $\mathcal{G}rp \downarrow \mathcal{G}rp$:

Take $\alpha = (\alpha_1, \alpha_2) : (P \xrightarrow{i_P} P * Q) \longrightarrow (G' \xrightarrow{\lambda'} H')$ and $\beta = (\beta_1, \beta_2) : (G \xrightarrow{\lambda} H) \longrightarrow (G' \xrightarrow{\lambda'} H')$ morphisms in $\mathcal{G}rp \downarrow \mathcal{G}rp$ with β_1 and β_2 surjective homomorphisms. P and Q are free groups so there exist $\varepsilon : P \longrightarrow G$ and $\phi'' : Q \longrightarrow H$ such that $\beta_1 \circ \varepsilon = \alpha_1$ and $\beta_2 \circ \phi'' = \alpha_2 \circ i_Q$. $\phi' = \lambda \circ \varepsilon$ and ϕ'' induce $\phi : P * Q \longrightarrow H$ verifying $\beta_2 \circ \phi = \alpha_2$. Finally the pair $(\varepsilon, \phi) : (P \xrightarrow{i_P} P * Q) \longrightarrow (G \xrightarrow{\lambda} H)$ is a morphism in $\mathcal{G}rp \downarrow \mathcal{G}rp$, because $\phi \circ i_P = \lambda \circ \varepsilon$ and satisfy $\alpha = \beta \circ (\varepsilon, \phi)$.

Since the forgetful functor $U_1 : \mathcal{PCM} \longrightarrow \mathcal{G}rp \downarrow \mathcal{G}rp$ (see Theorem 2.3) preserves surjective morphisms, its left adjoint $F_1 : \mathcal{G}rp \downarrow \mathcal{G}rp \longrightarrow \mathcal{PCM}$ preserves projective objects, and so $F_1(P \xrightarrow{i_P} P * Q)$ is a projective precrossed module. $\square$

Note that free crossed modules relative to $\nu : \mathcal{CM} \longrightarrow \mathcal{Set}$ [5] are obtained from free precrossed modules (relative to $\mathcal{U} : \mathcal{PCM} \longrightarrow \mathcal{Set}$) making its Peiffer abelianisation. In a similar way if we quotient a projective precrossed module by its Peiffer subgroup we get a projective crossed module.

A non zero free precrossed module $(\overline{F}, F * F, \langle i_1, Id \rangle_{|\overline{F}})$ is never a crossed module. In fact any projective precrossed module in the family of Proposition 2.5, $(\overline{P}, P * Q, \langle i_P, Id \rangle_{|\overline{P}})$ with $P \neq 0$ is not a crossed module. To see this, take a non trivial element $p \in P$ and then it is clear that $\langle i_P, Id \rangle(p)p \neq ppp^{-1} = p$.

Observe that every projective precrossed module (M, P, μ) is a retract of a free precrossed module, so M and P will be free groups.

3. COTRIPLE (CO)HOMOLOGY OF PRECROSSED MODULES

In this section we particularize to our context Barr and Beck's cotriple (co)homology [1]. Before introducing homology and cohomology theories for precrossed modules, we study the category of abelian group objects in the category of precrossed modules.

The category of abelian group objects in a category plays a fundamental role in the description of the homology and the cohomology of the category. It is shown in [5] that the abelian group objects in the category $\mathcal{CM}$ are just the abelian crossed modules (i.e. those which coincide with its centre), as the abelian group objects in the category $\mathcal{G}rp$ are simply the abelian groups.

PROPOSITION 3.1. *The forgetful functor $J : \mathcal{APCM} \longrightarrow \mathcal{PCM}$ is a full and faithful embedding whose replete image consists of abelian precrossed modules.*

Proof. Is an analogue to the one made in [5] for crossed modules, simply change $\mathcal{CM}$ by $\mathcal{PCM}$. $\square$

The inclusion J of the variety $\mathcal{APCM}$ in $\mathcal{PCM}$ has a left adjoint (the reflector) $ab : \mathcal{PCM} \longrightarrow \mathcal{APCM}$ termed the *abelianisation functor*, which assigns to a precrossed module (M, P, μ) the abelian precrossed module $(M, P, \mu)_{ab} = (M / [M, M] [P, M], P / [P, P], \bar{\mu})$.

The functors $\mathcal{F} : \mathcal{Set} \longrightarrow \mathcal{PCM}$ and $\mathcal{U} : \mathcal{PCM} \longrightarrow \mathcal{Set}$ described in section 2 induce a free cotriple (C, δ, ε) in $\mathcal{PCM}$, with $C = \mathcal{F} \circ \mathcal{U} : \mathcal{PCM} \longrightarrow \mathcal{PCM}$, $\delta : C \Rightarrow Id_{\mathcal{PCM}}$ counit of the adjunction and $\varepsilon : C \Rightarrow C^2$ the comultiplication. Every precrossed module (M, P, μ) has a *standard free simplicial resolution* $C.(M, P, \mu) \longrightarrow (M, P, \mu)$, where $C.(M, P, \mu)$ is the simplicial precrossed module with n -dimensional object $C_n(M, P, \mu) = C^{n+1}(M, P, \mu)$, for $n \geq 0$ and with face and degeneracy operators

$$d_i = C^{n-i} \delta C^i(M, P, \mu) : C_n(M, P, \mu) \longrightarrow C_{n-1}(M, P, \mu), 0 \leq i \leq n$$

$$s_i = C^{n-i} \varepsilon C^i(M, P, \mu) : C_n(M, P, \mu) \longrightarrow C_{n+1}(M, P, \mu), 0 \leq i \leq n$$

Applying the abelianisation functor $ab : \mathcal{PCM} \longrightarrow \mathcal{APCM}$ we obtain an augmented simplicial complex of abelian precrossed modules

$$(C.(M, P, \mu))_{ab} \twoheadrightarrow (M, P, \mu)_{ab}$$

If we take the alternating sum of the abelianised face operators we get the chain complex of abelian precrossed modules

$$\dots (C_n(M, P, \mu))_{ab} \xrightarrow{\partial_n} (C_{n-1}(M, P, \mu))_{ab} \dots \xrightarrow{\partial_1} (C(M, P, \mu))_{ab} \longrightarrow 0$$

whose homology provides the *homology groups* of the precrossed module (M, P, μ) :

$$H_n(M, P, \mu) = H_{n-1}((C.(M, P, \mu))_{ab}, \partial_*), \text{ for } n \geq 1$$

On the other hand, if (A, B, f) is an abelian precrossed module we apply the functor $Hom_{\mathcal{PCM}}(-, (A, B, f))$ to $C.(M, P, \mu)$ and we obtain an augmented simplicial complex of abelian groups

$$Hom_{\mathcal{PCM}}((M, P, \mu), (A, B, f)) \twoheadrightarrow Hom_{\mathcal{PCM}}(C.(M, P, \mu), (A, B, f))$$

and the chain complex of abelian groups

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_{\mathcal{PCM}}(C(M, P, \mu), (A, B, f)) \xrightarrow{\partial^1} \dots \\ \dots &\longrightarrow \operatorname{Hom}_{\mathcal{PCM}}(C_n(M, P, \mu), (A, B, f)) \xrightarrow{\partial^n} \\ &\xrightarrow{\partial^n} \operatorname{Hom}_{\mathcal{PCM}}(C_{n+1}(M, P, \mu), (A, B, f)) \dots \end{aligned}$$

We will define the *cohomology groups* of (M, P, μ) with coefficients in (A, B, f) for $n \geq 1$ as

$$\begin{aligned} H^n((M, P, \mu), (A, B, f)) &= H^{n-1}(\operatorname{Hom}_{\mathcal{PCM}}(C(M, P, \mu), (A, B, f)), \partial^*) \\ &= H^{n-1}(\operatorname{Hom}_{\mathcal{PCM}}((C(M, P, \mu))_{ab}, (A, B, f)), \partial^*) \end{aligned}$$

Note that the homology groups of a precrossed module are actually abelian group objects in $\mathcal{PCM}$.

PROPOSITION 3.2. *It is verified:*

- (i) $H_n(-) : \mathcal{PCM} \longrightarrow \mathcal{APCM}$ and $H^n(-, -) : \mathcal{PCM}^{op} \times \mathcal{APCM} \longrightarrow \mathcal{Ab}$ are functors for each $n \geq 1$.
- (ii) For every precrossed module (M, P, μ) and all abelian precrossed module (A, B, f)

$$H_1(M, P, \mu) = (M, P, \mu)_{ab}$$

$$\begin{aligned} H^1((M, P, \mu), (A, B, f)) &= \operatorname{Hom}_{\mathcal{PCM}}((M, P, \mu), (A, B, f)) = \\ &= \operatorname{Hom}_{\mathcal{PCM}}((M, P, \mu)_{ab}, (A, B, f)) \end{aligned}$$

- (iii) If (M, P, μ) is a projective precrossed module then $H_n(M, P, \mu) = 0 = H^n((M, P, \mu), (A, B, f))$ for each abelian precrossed module (A, B, f) and $n \geq 2$.

- (iv) If $0 \longrightarrow (A_1, B_1, f_1) \longrightarrow (A, B, f) \longrightarrow (A_2, B_2, f_2) \longrightarrow 0$ is exact, then there exists a long exact sequence in cohomology

$$\begin{aligned} \dots &\longrightarrow H^n((M, P, \mu), (A_1, B_1, f_1)) \longrightarrow H^n((M, P, \mu), (A, B, f)) \longrightarrow \\ &\longrightarrow H^n((M, P, \mu), (A_2, B_2, f_2)) \longrightarrow H^{n+1}((M, P, \mu), (A_1, B_1, f_1)) \dots \end{aligned}$$

Remark 1. An alternative way for computing the (co)homology is obtained replacing the standard free simplicial resolution by a *projective simplicial resolution* [1], that is, an augmented simplicial complex of precrossed modules $(M_\bullet, P_\bullet, \mu_\bullet) \longrightarrow (M, P, \mu)$ where (M_n, P_n, μ_n) is a projective precrossed module for each $n \geq 0$, and the simplicial set $\mathcal{U}(M_\bullet, P_\bullet, \mu_\bullet) = M_\bullet \times P_\bullet \longrightarrow \mathcal{U}(M, P, \mu) = M \times P$ has a contraction.

It is shown in [1] that $H_n(M, P, \mu) = H_{n-1}((M_\bullet, P_\bullet, \mu_\bullet)_{ab}, \partial_*)$ and $H^n((M, P, \mu), (A, B, f)) = H^{n-1}(\operatorname{Hom}_{\mathcal{PCM}}((M_\bullet, P_\bullet, \mu_\bullet)_{ab}, (A, B, f)), \partial^*)$ for each abelian precrossed module (A, B, f) and all $n \geq 1$.

Observe that $\mathcal{U}(M., P., \mu.) \longrightarrow \mathcal{U}(M, P, \mu)$ is obviously a Kan complex, so $\mathcal{U}(M., P., \mu.) \longrightarrow \mathcal{U}(M, P, \mu)$ has a contraction if and only if $M. \longrightarrow M$ and $P. \longrightarrow P$ are both weak equivalences, i.e. the homotopy groups verify $\pi_i(M.) = 0 = \pi_i(P.)$ for $i \geq 1$, and $\pi_0(M.) = M$, $\pi_0(P.) = P$.

4. RELATIONSHIPS WITH ANOTHER (CO)HOMOLOGY THEORIES

We begin with a theorem where the principal relationships between our (co)homology theory for precrossed modules and the Eilenberg-MacLane (co)homology groups are pointed out.

Denote by $H_n(G) = H_n(G, Z)$ the *nth integral homology group* of a group G , and by $H^n(G, A)$ the *nth cohomology group* of a group G with coefficients in an abelian group A (considered as a trivial G -module).

The category of groups is a subcategory of the category of precrossed modules by thinking of it as included by the functor $i : \mathcal{G}rp \longrightarrow \mathcal{PCM}$, $i(G) = (1, G, i)$. i has a right adjoint $k : \mathcal{PCM} \longrightarrow \mathcal{G}rp$, $k(M, P, \mu) = P$.

Another common way of thinking of a group as a precrossed module is through the functor $\varepsilon : \mathcal{G}rp \longrightarrow \mathcal{PCM}$, $\varepsilon(G) = (G, G, id)$, where G acts on itself by conjugation. ε has as left adjoint the functor k .

THEOREM 4.1. *(i) For every group G and every abelian precrossed module (A, B, f)*

$$H_n(iG) \cong iH_n(G), \quad H^n(iG, (A, B, f)) \cong H^n(G, B), \quad n \geq 1$$

(ii) For every precrossed module (M, P, μ) and all $n \geq 1$

$$kH_n(M, P, \mu) \cong H_n(P)$$

(iii) For every precrossed module (M, P, μ) and every abelian group A ,

$$H^n((M, P, \mu), \varepsilon A) \cong H^n(P, A)$$

for all $n \geq 1$.

Proof. As it was seen in [1], Eilenberg-MacLane (co)homology groups can be computed through free simplicial resolutions. That is, given a group G , if $F. \twoheadrightarrow G$ is an augmented simplicial group such that each F_n is a free group for every $n \geq 0$, and $\pi_i(F.) = 0$ for $i > 0$, $\pi_0(F.) = G$, then

$$H_n(G) \cong H_{n-1}((F.)_{ab}, \partial_*)$$

$$H^n(G, A) \cong H^{n-1}(Hom_{\mathcal{G}rp}((F.), A), \partial^*)$$

for every abelian group A and $n \geq 1$, where the differential is taken to be the alternating sum of the abelianised face operators.

To prove (i) take a free simplicial resolution $F. \twoheadrightarrow G$ of the group G . By Proposition 2.5, the augmented simplicial precrossed module $iF. \twoheadrightarrow iG$ is a projective simplicial resolution of the precrossed module iG . Then

$$\begin{aligned} H_n(iG) &\cong H_{n-1}((iF.)_{ab}) \cong H_{n-1}(i(F.ab)) \cong iH_{n-1}((F.)_{ab}) \cong iH_n(G), \quad n \geq 1 \\ H^n(iG, (A, B, f)) &\cong H^{n-1}(\text{Hom}_{\mathcal{PCM}}((iF.), (A, B, f))) \\ &\cong H^{n-1}(\text{Hom}_{\mathcal{GRP}}(F., k(A, B, f))) \cong H^{n-1}(\text{Hom}_{\mathcal{GRP}}(F., B)) \\ &\cong H^n(G, B), \quad n \geq 1 \end{aligned}$$

Observe that we used that i preserves kernels and cokernels and commutes with the abelianisation functors ($i \circ ab = ab \circ i$).

Now we prove (ii) and (iii). Recall that as it was seen at the end of last section, for every precrossed module (M, P, μ) the simplicial group $k(C.(M, P, \mu))$ is a free simplicial resolution of the group P . Then

$$\begin{aligned} kH_n(M, P, \mu) &= kH_{n-1}(C.(M, P, \mu)_{ab}) \cong H_{n-1}(k(C.(M, P, \mu))_{ab}) \\ &\cong H_n(P), \quad n \geq 1 \end{aligned}$$

since $k \circ ab = ab \circ k$.

Similarly, for every abelian group A we get

$$\begin{aligned} H^n((M, P, \mu), \varepsilon A) &= H^{n-1}(\text{Hom}_{\mathcal{PCM}}(C.(M, P, \mu), \varepsilon A)) \\ &\cong H^{n-1}(\text{Hom}_{\mathcal{GRP}}(kC.(M, P, \mu), A)) \cong H^n(P, A), \quad n \geq 1 \quad \square \end{aligned}$$

In [9], Gilbert introduces two invariants $\mathcal{H}_1(T, G, \partial)$ and $\mathcal{H}_2(T, G, \partial)$ for every crossed module (T, G, ∂) using the equivalence between the category of crossed modules and the category of group objects in the category of groupoids.

For each crossed module (T, G, ∂) , Gilbert's homology groups are the abelian crossed modules

$$\mathcal{H}_1(T, G, \partial) = (T/[G, T], G/[G, G], \bar{\partial})$$

$$\mathcal{H}_2(T, G, \partial) = (\Sigma, H_2(G), \sigma_{*|\Sigma})$$

where $\Sigma = \text{Ker}(\tau_*)$, τ_* and σ_* are the homomorphisms induced in homology by the group homomorphisms

$$\tau : T \rtimes G \longrightarrow G, \quad \tau(t, g) = g$$

$$\sigma : T \rtimes G \longrightarrow G, \quad \sigma(t, g) = \partial(t)g$$

and $T \rtimes G$ denotes the semidirect product of G acting on T .

THEOREM 4.2. *If (T, G, ∂) is a crossed module then $H_1(T, G, \partial) \cong \mathcal{H}_1(T, G, \partial)$ and $H_2(T, G, \partial) \cong \mathcal{H}_2(T, G, \partial)$.*

Proof. The first equality is clear because for each crossed module (T, G, ∂) it is verified that $[T, T] = [\partial(T), T] \subset [G, T]$.

To prove the second part of the theorem note that if $(\overline{F}, F * F, \langle i_1, Id \rangle_{|\overline{F}})$ is a free precrossed module then $\overline{F} \rtimes (F * F)$ is a free group since the short exact sequence $\overline{F} \hookrightarrow F * (F * F) \xrightarrow{\langle 0, Id \rangle} F * F$ is split, and the action induced by $F * F$ on $\overline{F}$ coincides with the action in the free precrossed module.

So if $C(T, G, \partial) = (M, P, \mu) \twoheadrightarrow (T, G, \partial)$ is the standard free simplicial resolution of (T, G, ∂) , then $M \rtimes P \twoheadrightarrow T \rtimes G$ is a free simplicial resolution of the group $T \rtimes G$ since its underlying augmented simplicial set coincides with $\mathcal{U}(M, P, \mu) \longrightarrow \mathcal{U}(M, P, \mu)$.

Let's take the group homomorphisms $\tau_n : M_n \rtimes P_n \longrightarrow P_n$, $\tau_n(m, p) = p$, and $\sigma_n : M_n \rtimes P_n \longrightarrow P_n$, $\sigma_n(m, p) = \mu_n(m)p$. It is easy to see that σ and τ are morphisms of augmented simplicial groups

$$\begin{array}{ccc} M \rtimes P & \twoheadrightarrow & T \rtimes G \\ \sigma \downarrow \tau & & \sigma \downarrow \tau \\ P & \twoheadrightarrow & G \end{array}$$

If we denote $\mathcal{H}_2(T, G, \partial) = (\Sigma, H_2(G), (\sigma_1)_* \Sigma)$ then $H_2(T \rtimes G) = H_1((M \rtimes P)_{ab})$, and $\Sigma = Ker((\tau_1)_*)$ where $(\tau_1)_*, (\sigma_1)_* : H_1((M \rtimes P)_{ab}) \longrightarrow H_1((P)_{ab})$ are induced in homology by $(\tau_1)_{ab}$ and $(\sigma_1)_{ab}$.

Since every epimorphism $M_n \rtimes P_n \xrightarrow{\tau_n} P_n$ is split, the short exact sequence of simplicial abelian precrossed modules

$$Ker(\tau)_{ab} \hookrightarrow (M \rtimes P)_{ab} \xrightarrow{(\tau)_{ab}} (P)_{ab}$$

is a weakly split sequence and the associated long exact sequence in homology provides the isomorphism

$$H_n(Ker(\tau)_{ab}) \cong Ker((\tau)_* : H_n((M \rtimes P)_{ab}) \longrightarrow H_n((P)_{ab}))$$

It is easy to prove that $[M_n \rtimes P_n, M_n \rtimes P_n] = [M_n, M_n] [P_n, M_n] \rtimes [P_n, P_n]$ so $(\tau_n)_{ab} : M_n / [M_n, M_n] [P_n, M_n] \times P_n / [P_n, P_n] \longrightarrow P_n / [P_n, P_n]$ is the projection to the second component, and we have a simplicial abelian group morphism

$$\begin{array}{ccc} Ker(\tau)_{ab} & \longrightarrow & Ker(\tau_{ab}) \\ (\sigma)_{ab} \downarrow & & \downarrow \sigma_{ab} \\ (G)_{ab} & \longrightarrow & G_{ab} \end{array} \quad (1)$$

In homology, between both of the first homology groups we obtain the abelian group homomorphism $\mathcal{H}_2(T, G, \partial)$.

Now observe that $(\sigma)_{ab|Ker(\tau)_{ab}} = (\mu)_{ab}$ and $\sigma_{ab|Ker(\tau)_{ab}} = \partial_{ab}$. It follows that the complex (1) is the abelianisation of the standard free simplicial resolution of (T, G, ∂) and so its first homology is $H_2(T, G, \partial)$ too. $\square$

Recall that the category of precrossed modules is equivalent to the category of simplicial groups of length 1 (*pre-cat¹-groups* in the notation of [4]) that is, a precrossed module (M, P, μ) is equivalent to the simplicial group

$$M \rtimes P \begin{array}{c} \xrightarrow{\tau} \\ \xleftarrow{\sigma} \end{array} P$$

where $\tau(m, p) = p$ and $\sigma(m, p) = \mu(m)p$.

Following the proof of Theorem 4.2 and replacing the crossed module (T, G, ∂) by a precrossed module (M, P, μ) we deduce:

THEOREM 4.3. *The n th homology group of a precrossed module (M, P, μ) is $H_n(M, P, \mu) = (\Sigma_n, H_n(P), \sigma_*)$, where $\Sigma_n = \text{Ker}(\tau_*)$ and $\tau_*, \sigma_* : H_n(M \rtimes P) \longrightarrow H_n(P)$ are the group homomorphisms induced in homology by τ and σ .*

COROLLARY 4.4. *If we think a group G as the precrossed module $(G, 1, 1)$ then $H_n(G, 1, 1) = (H_n(G), 1, 1)$.*

Example. Let G be a group regarded as the precrossed module (G, G, Id) . σ and τ composed with the isomorphism $G \times G \rightarrow G \rtimes G$ given by $(g_1, g_2) \mapsto (g_1 g_2^{-1}, g_2)$ become the projections from $G \times G$ to G . So Künneth formula in homology of groups and Theorem 4.3 yield that

$$H_2(G, G, \text{Id}) = (H_2(G) \oplus (H_1(G) \otimes H_1(G)), H_2(G), \sigma_*)$$

with σ_* acting as the identity on $H_2(G)$ and as zero on $H_1(G) \otimes H_1(G)$.

The category of crossed modules $\mathcal{CM}$ is a Birkhoff variety of $\mathcal{PCM}$. It is well known that the inclusion $I : \mathcal{CM} \longrightarrow \mathcal{PCM}$ has as left adjoint $\mathcal{P} : \mathcal{PCM} \longrightarrow \mathcal{CM}$ the Peiffer abelianisation which assigns to each precrossed module (M, P, μ) the quotient $(M/\langle M, M \rangle, P, \bar{\mu})$.

Consider the composition of adjunctions

$$\text{Set} \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \xleftarrow{\mathcal{U}} \end{array} \mathcal{PCM} \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \xleftarrow{I} \end{array} \mathcal{CM}$$

The tripleable forgetful functors $\mathcal{U}$ and $\mathcal{UI}$ provide the free cotriples (C, δ, ε) in $\mathcal{PCM}$ and $(\overline{C}, \delta, \varepsilon)$ in $\mathcal{CM}$ (this last cotriple is the one obtained in [5]).

To relate our (co)homology theory of precrossed modules with CCG -(co)homology theory of crossed modules we will define natural transformations

$$r_n : C^n I \Rightarrow I \overline{C}^n$$

inductively by $r_{n+1} = r_n \overline{C} \circ C^n r_1$, for every $n \geq 1$ being r_1 the natural transformation which sends each crossed module (T, G, ∂) to the canonical projection $C(T, G, \partial) \twoheadrightarrow \mathcal{P}C(T, G, \partial) = \overline{C}(T, G, \partial)$.

LEMMA 4.5. *The transformations r_n commute with the face operators of the standard free resolutions $C.(T, G, \partial)$ and $\overline{C}.(T, G, \partial)$, that is, $r_n \circ \delta_i = \overline{\delta}_i \circ r_{n+1}$, where $\delta_i = C^{n-i} \delta C^i$ and $\overline{\delta}_i = \overline{C}^{n-i} \overline{\delta} \overline{C}^i$.*

Proof. It is an easy calculation made in [6] in a more general context. $\square$

So for every crossed module (T, G, ∂) we have a surjective morphism of resolutions

$$\begin{array}{ccccccc} \dots & C^n(T, G, \partial) & \dots & C^2(T, G, \partial) & \xrightarrow[\delta_0]{\delta_1} & C(T, G, \partial) & \xrightarrow{\delta} (T, G, \partial) \\ & r_n \downarrow & & r_2 \downarrow & & r_1 \downarrow & \parallel \\ \dots & \overline{C}^n(T, G, \partial) & \dots & \overline{C}^2(T, G, \partial) & \xrightarrow[\overline{\delta}_0]{\overline{\delta}_1} & \overline{C}(T, G, \partial) & \xrightarrow{\overline{\delta}} (T, G, \partial) \end{array}$$

In the next theorem we denote the n th CCG -homology group of the crossed module (T, G, ∂) by $H_n^{CCG}(T, G, \partial)$ and the n th CCG -cohomology group of the crossed module (T, G, ∂) with coefficients in an abelian precrossed module (A, B, f) by $H_{CCG}^n((T, G, \partial), (A, B, f))$.

THEOREM 4.6. *For every crossed module (T, G, ∂) and each abelian precrossed module (A, B, f)*

$$(i) \ H_1(T, G, \partial) = H_1^{CCG}(T, G, \partial)$$

$$H^1((T, G, \partial), (A, B, f)) = H_{CCG}^1((T, G, \partial), (A, B, f))$$

(ii) *There exists a long exact sequence of abelian precrossed modules in homology*

$$\begin{aligned} \dots H_n^{CCG}(T, G, \partial) &\longrightarrow H_{n-2}(Ker(r.)_{ab}) \longrightarrow H_{n-1}(T, G, \partial) \\ &\longrightarrow H_{n-1}^{CCG}(T, G, \partial) \longrightarrow \dots \longrightarrow H_1(Ker(r.)_{ab}) \\ &\longrightarrow H_2(T, G, \partial) \longrightarrow H_2^{CCG}(T, G, \partial) \longrightarrow 0 \end{aligned}$$

(iii) *There exists a long exact sequence of abelian precrossed modules in cohomology*

$$\begin{aligned} 0 &\longrightarrow H_{CCG}^2((T, G, \partial), (A, B, f)) \longrightarrow H^2((T, G, \partial), (A, B, f)) \\ &\longrightarrow H^1(Coker(Hom_{\mathcal{PCM}}(r., (A, B, f)))) \dots H_{CCG}^n((T, G, \partial), (A, B, f)) \\ &\longrightarrow H^n((T, G, \partial), (A, B, f)) \longrightarrow H^{n-1}(Coker(Hom_{PCM}(r., (A, B, f)))) \\ &\longrightarrow H_{CCG}^{n+1}((T, G, \partial), (A, B, f)) \longrightarrow \dots \end{aligned}$$

Proof.

(i) it is clear since $H_1^{CCG}(T, G, \partial) = (T, G, \partial)_{ab}$ and $H_{CCG}^1((T, G, \partial), (A, B, f)) = \text{Hom}_{\mathcal{PCM}}((T, G, \partial), (A, B, f))$.

To prove (ii) note that there exists a surjective morphism of complexes

$$\begin{array}{ccccccc} \dots & C^n(T, G, \partial)_{ab} & \dots & C^2(T, G, \partial)_{ab} & \xrightarrow{\partial_2} & C(T, G, \partial)_{ab} & \longrightarrow 0 \\ & (r_n)_{ab} \downarrow & & (r_2)_{ab} \downarrow & & (r_1)_{ab} \downarrow & \\ \dots & \overline{C}^n(T, G, \partial)_{ab} & \dots & \overline{C}^2(T, G, \partial)_{ab} & \xrightarrow{\overline{\partial}_2} & \overline{C}(T, G, \partial)_{ab} & \longrightarrow 0 \end{array}$$

and $(r_1)_{ab}$ is the identity map because the abelianisation (as crossed module) of the Peiffer abelianisation of a precrossed module is exactly the abelianisation of the precrossed module. So the sequence follows from the long exact sequence in homology associated to the short exact sequence of complexes

$$\text{Ker}(r.)_{ab} \hookrightarrow C.(T, G, \partial)_{ab} \xrightarrow{(r.)_{ab}} \overline{C}.(T, G, \partial)_{ab}$$

In a parallel way is proved (iii) using

$$\begin{aligned} \text{Hom}_{\mathcal{PCM}}(C(T, G, \partial), (A, B, f)) &= \text{Hom}_{\mathcal{PCM}}((C(T, G, \partial))_{ab}, (A, B, f)) \\ &= \text{Hom}_{\mathcal{PCM}}(\overline{C}(T, G, \partial), (A, B, f)) \quad \square \end{aligned}$$

Remark 2. For $n \geq 2$, $H_n(G, G, Id) \neq H_n^{CCG}(G, G, Id) = (H_n(G), H_n(G), Id)$.

Remark 3. The previous results and Theorem 4.3 show that the low-dimensional homology given by Gilbert in [9] is not of crossed modules but of precrossed modules.

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DANIEL ARIAS, DEPARTAMENTO DE ALGEBRA, UNIVERSIDAD DE SANTIAGO, E-15782, SPAIN
E-mail address: `daniarmo@worldonline.es`

MANUEL LADRA, DEPARTAMENTO DE ALGEBRA, UNIVERSIDAD DE SANTIAGO, E-15782, SPAIN
E-mail address: `ladra@usc.es`

ALFREDO R-GRANDJEÁN, DEPARTAMENTO DE ALGEBRA, UNIVERSIDAD DE SANTIAGO, E-15782, SPAIN
E-mail address: `granjean@usc.es`